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HOW MANY SLOPES IN A POLYGON?

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ABSTRACT

The compactness theorem for the predicate calculus is used to prove that if p is a prime and s is a positive integer with $s \leqq 11$ or $2^{s-1} \leqq p$ and X is a set of distinct residues mod p, there are at least $2s-3$ distinct residues $x+y$ with $x \neq y$ and $x, y \in X$.

Consider an arithmetic progression of length s. It is plain that exactly $sk - k^2 + 1$ numbers can be written as the sum of k distinct elements of the progression. Furthermore, in any ordered abelian group, any s-element set has at least $sk - k^2 + 1$ such sums. This can easily be proven by induction — just remove the largest element from an $(s+1)$-element set. P. Erdös has conjectured that the same theorem holds in every $\mathbf{Z}_p$ for p prime and $\geqq sk - k^2 + 1$ [1].

For fixed k and s we can reformulate this conjecture in the form

$$\forall p \geqq sk - k^2 + 1 \qquad \mathbf{Z}_p \models \varphi_{k,s}$$

where $\varphi_{k,s}$ is a first order sentence of group theory. For instance, $\varphi_{2,3}$ is

$$\begin{aligned}
&\forall x_1, x_2, x_3\ \exists y_1, y_2, y_3[(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)\\
&\rightarrow (y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge (y_1 = x_1 + x_2 \vee y_1 = x_1 + x_3\\
&\vee y_1 = x_2 + x_3) \wedge (y_2 = x_1 + x_2 \vee y_2 = x_1 + x_3 \vee y_2 = x_2 + x_3)\\
&\wedge (y_3 = x_1 + x_2 \vee y_3 = x_1 + x_3 \vee y_3 = x_2 + x_3))].
\end{aligned}$$

As we have noted above, $\varphi_{k,s}$ is true in all ordered abelian groups. But every torsion abelian group can be ordered. Therefore $\varphi_{k,s}$ is true in all torsion free abelian groups. Thus by an application of the compactness theorem

THEOREM 1. *For fixed k and s, Erdös conjecture is true for all sufficiently large primes p.*

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For the rest of this paper we shall consider only the case $k = 2$. We shall prove that in this case the conjecture is true whenever $s \leqq 11$ or $p \geqq 2^{s-1}$.

In organizing the mind, one picture is worth a thousand words, so let us picture $\mathbf{Z}_p$ as the vertices of a regular p-gon. It now follows that two pairs $\{s, t\}$ and $\{u, v\}$ have the same sum iff the lines st and uv are parallel. (For instance, if we put 0 at the top, then the pairs summing to 0 are the horizontal lines.) Thus the case $k = 2$ would be settled if we knew all ways of choosing s points on a circle so that the lines joining them have a total of at most $2s - 4$ different slopes. One way of doing this is trivial. The regular n-gon has exactly n different slopes. So for instance any 7 vertices of a regular decagon have exactly ten slopes. This can be seen by a pigeonhole argument. A given slope has either 4 chords and 2 tangents or 5 chords, so in choosing 7 of the 10 vertices, one of these chords must have both endpoints selected. Thus all ten slopes will be represented. These trivial examples can not lead to a counterexample to Erdös conjecture, and so we will ignore them for the rest of this paper.

Let us turn to the cases $s \leqq 5$. $s = 1$ and 2 is trivial. For $s = 3$, any triangle must have 3 slopes. $s = 4$ is slightly more complicated (Fig. 1). Lines AB, AC, AD,

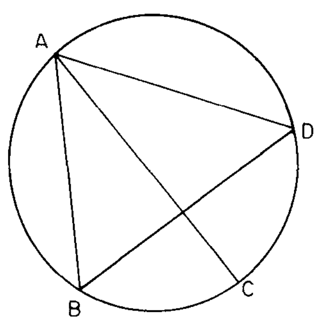

Fig. 1.

and BD all cross. So in order to have no more than 4 slopes, BC$\|$AD and CD$\|$AB. Thus, since the rectangle is the only parallelogram which can be inscribed in a circle, the figure must be a rectangle. No rectangle can be embedded into any prime-sided regular polygon, so Erdös conjecture holds for $s = 4$. Let us note that there is a whole continuous family of rectangles embeddable in a circle. Thus the conditions BC$\|$AD and CD$\|$AB do not uniquely determine the figure.

Now let us consider $s = 5$. There are two cases: either some face is parallel to another face or no two faces are parallel. We do the first case first. Figure 2 can be drawn, where AB$\|$CD. These lines have six different slopes. Thus CB$\|$DE

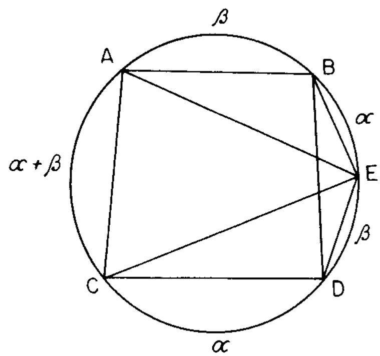

Fig. 2.

and AD ∥ BE. Now AB ∥ CD implies arc AC = arc BE + arc ED; CB ∥ DE implies arc CD = arc BE; AD ∥ BE implies arc DE = arc AB. Thus we may label Fig. 2 with the arcs α and β as shown. Now AC ∥ BE is impossible since it implies $\beta = \alpha + \beta$, $\alpha = 0$. Similarly AC is not parallel to DE, therefore AC ∥ BD and $\alpha = \beta$, and we have drawn 5 of the 6 vertices of a regular hexagon.

Now for the second case; no two faces are parallel. The figure is drawn in Fig. 3.

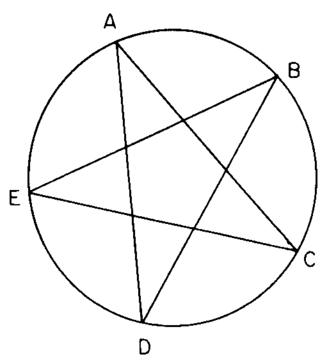

Fig. 3.

All five of these lines cross so at most one face can fail to be parallel to the opposite diagonal, say AE. Then DE ∥ AC, CD ∥ BE, BC ∥ AD, and AB ∥ CE. Consequently arc AE = arc DC, arc DE = arc BC, arc CD = arc AB, arc AE = arc BC. Thus the figure is a regular pentagon, and we see that $s = 5$ has no nontrivial cases.

Let us now turn to the case $s = 6$. To carry out this case, I was forced to rely on the vilage idiot, a computer. Since the ideas will be used again, let us go into some detail. Start with six points and fifteen lines (Fig. 4). Ignoring optimiza-

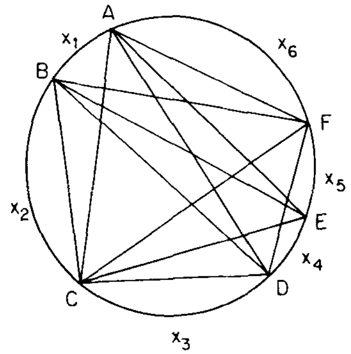

Fig. 4.

tions, we can say that the computer enumerated all possible equivalence relations on this set of fifteen lines which had no more than eight equivalence classes and which satisfied the condition that no two crossing lines were in the same equivalence class. Each such relation must be checked geometrically. This was done as follows. Assume the circle has circumference one and let the six arcs be labeled $x_1, x_2, x_3, x_4, x_5, x_6$ as shown. Then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$. Any pair of parallel lines leads to a further equation. For instance BC $\|$ AD iff $x_1 = x_3$ and BC $\|$ AF iff $x_1 = x_3 + x_4 + x_5$. The computer wrote out all the equations for each possible equivalence relation and solved them.[†] In almost all cases the system was either inconsistent or did not have any positive definite solutions, or had unique solutions which were all multiples of either 1/6, 1/7 or 1/8 thus leading to a trivial figure. The sole exceptions corresponded to Fig. 5, where ABDC is a rectangle and arc AF = arc FC = arc DE = arc BE. Since this figure contains a rectangle, it can not be embedded into any prime-sided regular polygon and so Erdös conjecture is true for $s = 6$. Note again that this is actually a one parameter family of figures so that the equivalence relation does not

[†]The method of solving the equations was somewhat specialized. The equation set was passed through three times. On the first pass all equations of the form $x = y$ were discharged. On each of the next two passes, all equations of the form $x = ay + bz$ (or $x = ay + bz + cu$, etc.), where a, b are positive integers, was discharged. At any time if an equation of the form $ax + by = 0$ ($ax + by + cz = 0$, etc.), where a, b are positive integers, is found, the case is terminated. If $0 = 1$ is found the case is also terminated. If after two passes all variables except one are gone and we are left with $ax = 1$ where a is a positive integer $\leqq 2s - 3$, the case is dismissed as trivial. In all other situations, all relevant information was sent to the printer and checked by hand. In all the hundreds of millions of cases done by the computer, this procedure was adequate to completely solve the equations set. The program was first written and extensively checked in Level II Basic. The case $s = 8$ required $3\frac{1}{2}$ weeks on a TRS-80. The program was then recoded into PLI. Extensive optimizations were added. The case $s = 8$ was finally reduced to 104 sec using PLIXCG. Both computers produced identical results for $s \leqq 8$.

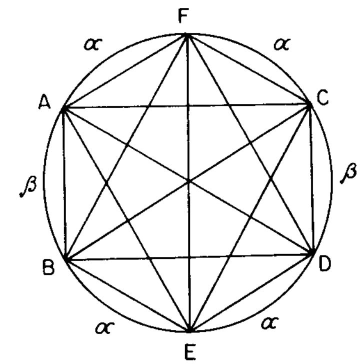

Fig. 5.

uniquely determine the figure. Alternatively the linear equation system does not have a unique solution.

The same thing was done for $s = 7$. The only resulting figure is shown in Fig. 6. Here we have not drawn all the lines. ACEG is again a rectangle so Erdös

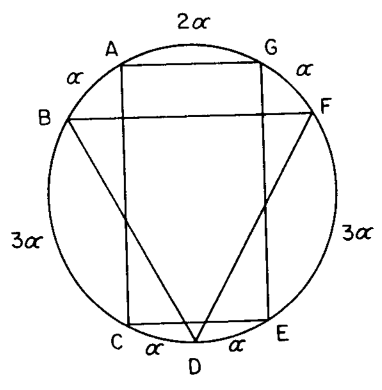

Fig. 6.

conjecture is true for $s = 7$. However ACEG has arc proportions two to one and the figure corresponds to 7 of the vertices of the regular 12-gon. It is also represented by the residues $\{0, \pm 1, \pm 4, \pm 5\}$ mod 12. Note here that the figure *is* uniquely determined by the equivalence relation. In other words the corresponding linear equation system has a unique solution.

DEFINITION. A set of points on a circle is *unproductive* if there are exactly s points but no more than $2s - 4$ different slopes. It is *minimal* unproductive if it is unproductive but has no unproductive proper subset.

LEMMA. *Let G be a torsion-free abelian group and let X be a minimal unproductive subset of $G \times \mathbf{Z}_n$ (i.e., a minimal set with s elements but $\leqq 2s-4$ sums). Then provided $s \geqq 5$ there is a $g \in G$ with $X \subseteq \{g\} \times \mathbf{Z}_n$.*

PROOF. Suppose otherwise. In order to draw a contradiction it suffices to show that X has a proper subset Y with k elements such that $X - Y$ has at least $2k$ fewer sums than X. To do this let G be ordered and choose the largest g such that $\{g\} \times Z_n$ has a non-empty intersection with X. Let $Y = (\{g\} \times Z_n) \cap X$. There are now three tiresome cases according to whether $k = 1$, $k = 2$ or $k \geqq 3$. These are entirely routine and are left to the reader.

LEMMA. *For fixed s and n the same proposition is true when G is replaced by any sufficiently large $\mathbf{Z}_p$.*

PROOF. Suppose otherwise that there is an increasing sequence $\langle p_i : i \in \omega \rangle$ and $X_i \subseteq \mathbf{Z}_{p_i} \times \mathbf{Z}_n$ such that X_i is an s-element minimal unproductive subset of $\mathbf{Z}_{p_i} \times \mathbf{Z}_n$ not contained in any one coset of $\mathbf{Z}_n$. Let $G = \Pi \mathbf{Z}_{p_i} / \mathcal{U}$ for $\mathcal{U}$ a non-principal ultrafilter on ω. Then $\Pi (\mathbf{Z}_{p_i} \times \mathbf{Z}_n)/\mathcal{U}$ is isomorphic to $G \times \mathbf{Z}_n$ and $X = \Pi X_i / \mathcal{U}$ would also have exactly s elements. Thus it is easily seen that the previous lemma leads to a contradiction.

THEOREM 2. *Any minimal unproductive set of points on a circle with $\geqq 5$ elements is uniquely determined by its equivalence relation.*

PROOF. Let $P_1, \cdots, P_s$ be the minimal unproductive set in counterclockwise order. As above let x_i be the arc from P_i to P_{i+1}. We may as well assume $x_1 + x_2 + \cdots + x_s = 1$. For each pair of lines there is a linear equation such that the lines are parallel iff the equation holds. So along with our unproductive set there is a system of linear equations. We shall prove the system has a unique solution. Using Gaussian elimination, there are $\{y_1, \cdots, y_k\} \leqq \{x_1, \cdots, x_s\}$ and rational numbers a_i and b_{ij} such that

$$x_i = a_i + \sum_{j=1}^{k} b_{ij} y_j$$

describes all solutions to the system. We must show that all the b_{ij} are zero. For $y'_1, \cdots, y'_k$ any set of parameters, define $x'_i = a_i + \Sigma_j b_{ij} y'_j$. And $P'_1 = P'_1$, $P'_{i+1} = P'_i + x'_i$ (mod 1). Clearly if the y'_j are close enough to y_j, then the points P'_i will still be distinct, in counterclockwise order, and have exactly the same parallels as the P_i.

Let n be the least common denominator of the a_i and b_{ij}. Now choose p so large that:

(1) the previous lemma holds for these values of n, s, p;

(2) the y'_j may be chosen to be multiples of n/p with the P'_i being distinct, in counterclockwise order and having exactly the same parallels as the P_i, and so that for every i, if $b_{i1}, b_{i2}, \cdots, b_{ik}$ are not all zero then $\Sigma_{j=1}^{k} b_{ij} y'_j \neq 0$.

Clearly such a p exists. The $P'_1, \cdots, P'_s$ must also be a minimal unproductive set. But each x'_i is of the form $a'_i/n + b'_i/p$ where a'_i and b'_i are integers. Thus the P'_i can be regarded as a minimal unproductive subset of $\mathbf{Z}_p \times \mathbf{Z}_n$ and thus are contained in a single coset of $\mathbf{Z}_n$. The x'_i were just differences between p'_i's, so all x'_i are in $\mathbf{Z}_n$. That is to say all b'_i are zero. By condition (2) this means that all the b_{ij} are zero. This is what was to be shown.

Corollary 3. *Erdös conjecture is true for* $s \leqq 11$.

Proof. We need only consider the equivalence relations whose linear equations system has a unique solution. Again by a somewhat tedious calculation† these can be enumerated. For $8 \leqq s \leqq 11$ there are three possible configurations. One is the ten residues mod 18,

$$\{0, 1, 2, 6, 7, 8, 11, 12, 13, 17\},$$

and the others are the two eleven residue systems mod 20,

$$\{0, \pm 1, \pm 4, \pm 5, \pm 8, \pm 9\}, \qquad \{0, \pm 3, \pm 4, \pm 5, \pm 7, \pm 8\}.$$

In the first $\{17, 2, 8, 11\}$ forms a rectangle with arc-proportions two to one, and in the second $\{\pm 1, +9\}$ form a rectangle with arc-proportions four to one. Neither case can be embedded in a prime sided regular polygon. Note that the cases $s = 7$ and $s = 10$ are the first two cases of an infinite class. If s is of the form $3n + 4$ there is an unproductive, uniquely determined set with $2s - 4$ sums among the residues mod $2s - 2$. The general pattern can be inferred from $s = 10$ (see Fig. 7). The case $s = 11$ is also undoubtedly part of an infinite class, but without seeing the next case, I have not tried to figure out what it is. Do note, however, that the third example is obtained from the second by multiplying with the unit 3. The second example has the picture shown in Fig. 8. Note that it is the union of the counter example for $s = 6$ with a pentagon.

Corollary 4. *Erdös conjecture is true whenever* $p \geqq 2^{s-1}$.

† The calculation for $s = 11$ required about 40 hours of CPU time on an IBM 370. I wish to thank the Penn State University Computation Center for the cooperation they have shown me on this project.

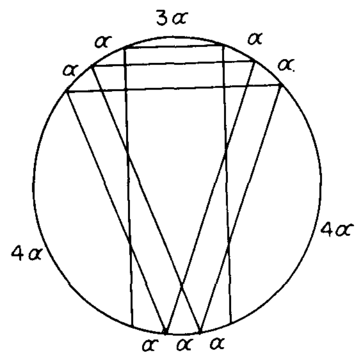

Fig. 7.

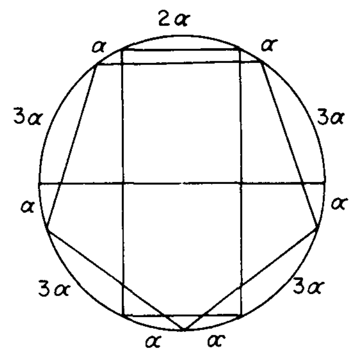

Fig. 8.

PROOF.[†] It suffices to show that the determinant of the above linear equation system has absolute value $<2^{s-1}$. This is so because if d is the determinant, the solution set can be written with denominator d and so is included in the d-sided regular polygon. Choosing s linearly independent rows, we see that there is one row consisting entirely of ones; the other rows all consist of one block of one's and one block of minus one's separated by zero's. (At least this is true when the rows are viewed cyclically.) Now perform column operations. Subtract the second column from the first, the third from the second, etc. We are left with a matrix having one row which is all zero except for a one in the last column, and all other rows having at most two ones and two minus ones in columns other than the last. By an initial column permutation we can see to it that at least one of these rows has at most three non-zero entries. We now expand by cofactors. The new matrix has one row of Euclidean norm at most $\sqrt{3}$, all others of norm at most 2. Recall Hadamard's Theorem that a determinant has absolute value equal to the volume of the parallelepiped formed from the row vectors. Thus the absolute value of the determinant is $\leqq$ the product of the norms of the rows. In our case this means that the determinant has absolute value $\leqq 2^{s-2}\cdot\sqrt{3}$ which is $<2^{s-1}$.

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[†] J. E. Olson has since proven a similar theorem with bound 4^{s-1} by completely different techniques.